Analysis of Students Action Proof when Making Counter-example on the Relationship between Area and Perimeter

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Abstract*. This study examines the process of proving mathematical actions through counter-example stimulation for elementary school students with different levels of problem-solving abilities. The study breaks down the process into three stages: initially creating conjectures, then confronting counter-examples, and finally re-evaluating the conjectures and providing evidence. The research adopts a qualitative approach using a case study method, with two out of 17 students being selected for analysis. The data analysis includes three stages: data reduction, data presentation, and conclusion drawing. The findings reveal that in the first stage, both the low and high level students' initial conjectures were incorrect. However, in the second stage, both groups improved their conjectures and proofs when confronted with counter-examples. Finally, in the third stage, the students' conjectures and proofs were found to be comprehensive. Overall, the study demonstrates the effectiveness of counter-example stimulation in supporting students' mathematical reasoning and problem-solving skills.*
Keywords – *Counter-example, Manipulative Objects, Area and Perimeter of a Rectangle.*

Abstrak. Penelitian ini menganalisis tahapan pembuktian tindakan melalui stimulasi *counter-example* pada siswa dengan tingkat kemampuan menyelesaikan soal yang tinggi dan rendah di sekolah dasar. Tahap pembuktian aksi dalam penelitian ini menggunakan tiga tahap: membuktikan dugaan primitif mereka, menghadapi contoh penyangkal, dan memeriksa kembali dugaan dan bukti. Jenis penelitian yang digunakan adalah kualitatif dengan pendekatan studi kasus. Subjek penelitian ini adalah dua dari 17 siswa sekolah dasar SDN Sidodadi yang dipilih secara *purposive sampling*. Teknik analisis data terdiri dari tiga tahap yakni reduksi data, penyajian data, dan penarikan kesimpulan. Hasil analisis menunjukkan bahwa pada tahap membuktikan dugaan primitif mereka, dugaan yang dibuat oleh siswa dengan tingkat rendah dan tinggi masih salah. Pada tahap mengkonfrontasikan *counter-example*, dugaan dan pembuktian yang dilakukan oleh siswa dengan tingkat rendah dan tinggi mengalami peningkatan. Pada tahap memeriksa kembali dugaan dan pembuktian, dugaan dan pembuktian yang dilakukan oleh siswa sudah komperehensif.
Kata Kunci – *Contoh-kontra, Benda Manipulatif, Luas dan Keliling Persegi Panjang*

I. INTRODUCTION

One of the fundamental concepts in mathematics is the relationship between the area and perimeter of a geometric shape. A deep understanding of this relationship not only requires mastery of the correct formulas but also the ability to formulate a strong conceptual understanding [1]. Mathematics is studied at nearly every level of education, which reflects the recognition that mathematics is essential for the development of knowledge and daily life. Schools are one of the institutions that provide students with the opportunity to learn, appreciate, and acquire value in mathematics, but many challenges hinder students from succeeding in these aspects [2].

Problem-solving skills can be implemented in all subjects, including mathematics. Mathematical problem-solving skills involve students' ability to solve contextual problems that are closely related to real-life applications of mathematical concepts [3]. Proof problems in mathematics often present unique challenges for students, especially when involving the use of manipulatives [4]. Some students with high levels of ability may struggle with the complexity of proofs, while those with lower levels of ability may encounter even greater difficulties. Counter-examples have the potential to assist elementary students in constructing more comprehensive mathematical conjectures [1].

In this context, the relationship between the area and perimeter of geometric shapes is a fundamental concept in mathematics. Understanding this relationship requires not only memorizing formulas but also grasping the basic principles of geometry. A deep understanding of the relationship between area and perimeter has significant implications in various mathematical contexts and everyday life. Students with strong conceptual understanding can more easily solve geometry problems, identify patterns, and formulate more precise mathematical solutions [5]. However, during the learning experience, students often encounter difficulties in fully comprehending this concept. Some concepts related to the relationship between area and perimeter may seem abstract and difficult for some students to grasp [6].

The difficulties experienced by students include making errors in calculations, which occur due to a lack of attention to detail and weaknesses in multiplication [7]. Difficulties are also frequently encountered by students in solving word problems in mathematics [8]. Some of the challenges faced by students include a lack of understanding of the steps required to solve word problems [9]. To improve their understanding, a teaching approach that involves proof through action, specifically through the creation of counter-examples, emerges as a promising strategy.

One type of informal proof at the elementary school level is action proof. Action proof at the elementary level is an instructional method that allows students to develop their basic mathematical skills [10]. Action proof is important for elementary students as it lays the foundation for achieving formal proof [11]. The role of action proof is to verify the truth of a mathematical statement with the help of manipulatives in the form of physical objects [12]. Action proof through object manipulation helps introduce proof-related problems at an early stage of mathematics learning for elementary students.

A counter-example is a method of mathematical proof used to determine whether a conjecture is true or false [13]. A counter-example provided by students can help them evaluate and refine the accuracy of their mathematical conjectures [14]. Additionally, counter-examples assist students in gaining a better understanding of the mathematical concepts being studied. Counter-examples are an essential component in teaching and learning mathematics, as they verify statements that can alter thinking or work towards finding better solutions. Therefore, the difficulty elementary students face in performing action proof lies in their failure to create logical mathematical conjectures when manipulating objects.

In this study, counter-examples are useful for stimulating students' mathematical conjectures when manipulating physical objects in action proof. The goal of using counter-examples is to make students aware of the errors in their mathematical conjectures and to correct them, leading to more comprehensive conjectures that prove the truth of a mathematical statement [5]. Problem-solving is one of the skills students must master after learning mathematics [15].

Based on the literature review and previous research, the stages of action proof using manipulatives through counter-example stimulation to enhance elementary students' mathematical conjectures are important for further analysis. The difficulties and failures of elementary students in making mathematical conjectures require additional study. These conjectures need to be improved through counter-example stimulation during object manipulation in action proof so that students' proof skills can reach the level of formal proof. This study aims to analyze various stages in the process of action proof through the use of counter-example stimulation. It is expected that this research will provide deeper insights into the differences in how students with varying ability levels understand and solve problems, as well as how counter-example stimulation can assist them in this process.

II. Methods

This study is a qualitative research that employs a descriptive qualitative approach with a case study design. A case study is a research method focused on a specific case, conducted with detailed, sharp, and in-depth processes to gain a comprehensive understanding of the case being studied [16]. n this research, the case study was conducted by analyzing two fifth-grade elementary school students. Two out of 17 fifth-grade students at SDN Sidodadi in the even semester of the 2023-2024 academic year were selected using purposive sampling techniques. The criteria for the subjects in this study were based on their ability to solve problems related to the perimeter and area of shapes. From the entire class, one student was chosen from each of two categories: low and high ability. The subjects were selected through several stages involving face-to-face interactions and collaboration with the class teacher to ensure the subjects were purposively representative.

Table 1. Categories of Problem-Solving Ability Levels

Score	**Ability Level**	**Number of Students**	**Subject (*Na*)**
$0 \leq Na \leq 60$	Low	2	*S1* (56)
$60 \leq Na \leq 80$	Medium	13	*S0* (70)
$80 \leq Na \leq 100$	High	2	*S2* (90)

The research instruments consisted of questions about the perimeter and area of rectangles, proof tasks, and manipulative objects. The proof tasks were adapted from Widjaja and Vale (2021), focusing on mathematical statements about the perimeter and area of rectangles, which required verification of their truthfulness. The statement was: “When you increase the area of a rectangle, the perimeter always increases. Explain why this is true or whether it is true? Is this statement true for all cases?” [1] We adapted Widjaja & Vale’s statement by using concrete objects. The concrete objects we used were manipulatives such as origami paper to find the area and string to find the perimeter of the rectangle. A series of questions were formulated based on the indicators of the proof stages through stimulation by counter-example.

Table 2. Indicators of Proof Actions Through Counter-Example Stimulation [17]

Stages	Indicators
Proving their primitive conjectures	Using object manipulation to make primitive conjectures
	Proving the truth of the given mathematical statement by creating and proving their primitive conjectures
Confronting the counter-example	Using manipulative objects to respond to the counter-example
	Proving the given counter-example
	Making new conjectures based on the given counter-example
Re-examining conjectures and proofs	Using object manipulation to find a more comprehensive conjecture
	Finding a new, more comprehensive conjecture from the proven counter-example

The data collection techniques consisted of tasks and documentation conducted through face-to-face interactions. The tasks involved giving proof questions to the students selected as subjects and recording them while they worked on the proof tasks. The data analysis technique consisted of three stages: data reduction, data presentation, and conclusion drawing [18]. The students' work data was reduced by separating out data that was not related to the indicators of proof actions through counter-example stimulation, as presented in Table 2. Data presentation was carried out by representing the subject's proof actions in the form of images. The data was then analyzed and organized descriptively based on the stages of proof actions. Meanwhile, conclusions were drawn by comparing the consistency of the data with triangulation and relevant theoretical analysis. Triangulation conclusions were provided when there was consistency between the images of the students' proof actions and the stages of the proof actions.

III. Results

The use of object manipulation in counter-example stimulation for two fifth-grade elementary school subjects (S1 and S2) was analyzed through three stages: proving their primitive conjectures, confronting the counter-example, and re-examining conjectures and proofs. In the stage of proving their primitive conjectures, S1 and S2 were presented with two rectangles, one with a length of 6 cm and a width of 2 cm, and the other with a length of 7 cm and a width of 3 cm. The first rectangle had an area of 12 cm² and a perimeter of 16 cm, while the second rectangle had an area of 21 cm² and a perimeter of 20 cm. To prove this conjecture, S1 created a new primitive conjecture by calculating two rectangles, the first with a length of 9 cm and a width of 3 cm, and the second with a length of 10 cm and a width of 4 cm. As shown in Figure 1, S1 represented these with manipulative objects by placing a string to measure the perimeter of the rectangle and origami paper to measure the area of the rectangle, all laid out on manila paper that had been measured with one square unit.

During the stage of finding the perimeter and area, S1 had difficulty placing the string and origami paper to match the length and width of the new primitive conjecture. As seen in Figure 1, when placing the origami paper and string downward, S1 added one square unit downward because they assumed that one square unit above was part of the length. This error resulted in an increase in the width, which should have yielded an area of 27 cm² and a perimeter of 24 cm for the first rectangle and an area of 40 cm² and a perimeter of 28 cm for the second rectangle. However, due to the incorrect placement, the results were an area of 36 cm² and a perimeter of 26 cm for the first rectangle and an area of 50 cm² and a perimeter of 30 cm for the second.

The primitive conjecture made by S1 was incorrect. S1 confirmed the statement given in the task sheet that when the area is increased, the perimeter will always increase. Additionally, S1 was unable to place the manipulative objects according to the length and width of their primitive conjecture. However, they did successfully represent the shape of the rectangle. On the other hand, S2 provided a primitive conjecture for two rectangles, the first with a length of 5 cm and a width of 3 cm, and the second with a length of 7 cm and a width of 5 cm. As shown in Figure 2, S2 created a representation using manipulative objects in the form of string and origami paper, placing them on the manila paper. S2 was able to place the string along the 5 cm length and 2 cm width according to the first rectangle's dimensions. Similarly, for the second rectangle, S2 correctly placed the string and origami paper along the 7 cm length and 5 cm width. The results matched their primitive conjecture.

The perimeter and area results for the rectangles in Figure 2 also corresponded with the calculations using the perimeter and area formulas for rectangles. The primitive conjecture made by S2 was incorrect. S2 confirmed the statement provided in the task sheet that when the area of a rectangle is increased, the perimeter will always increase. However, S2 successfully placed the manipulative objects according to the rectangle's length and width and represented the string and origami paper as a rectangle.

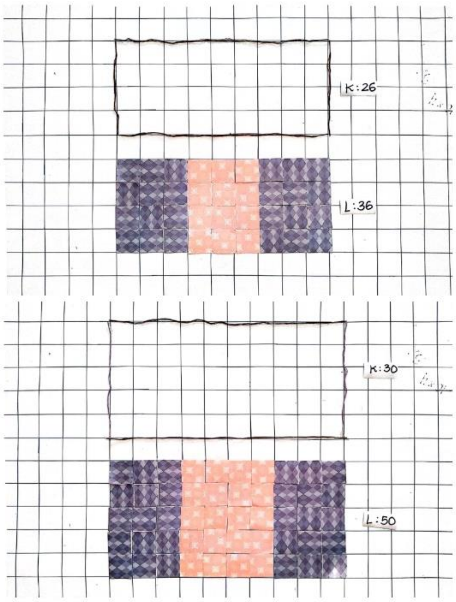

Figure 1. *S1's Primitive Conjecture*

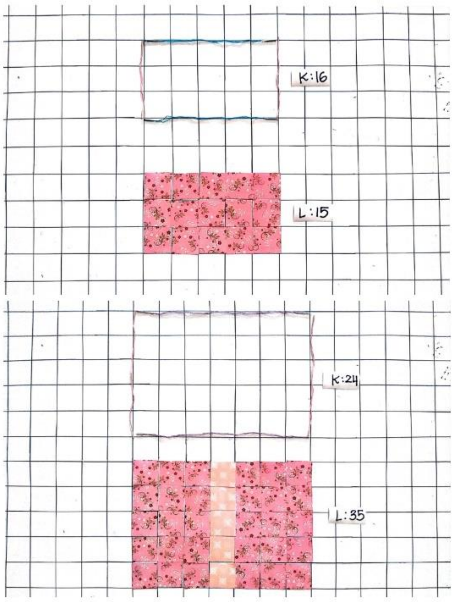

Figure 2. S2's Primitive Conjecture

Counter-Example Faced

The initial conjectures provided by S1 and S2 were still inaccurate, necessitating a counter-example to obtain a more in-depth analysis. We introduced a counter-example using two rectangles, the first with a length of 5 cm and a width of 2 cm, and the second with a length of 4 cm and a width of 3 cm. S1 and S2 were asked to make conjectures using manipulative objects. At this stage, S1 still had some difficulty placing the string and origami paper according to the length and width of the rectangle. We assisted S1 in placing the string and origami paper according to the rectangle's dimensions. S1 began by placing the string horizontally with a length of 5 cm and then vertically with a length of 2 cm without adding an extra square unit to find the perimeter. Then, S1 placed origami paper covering 5 square units horizontally and 2 square units vertically to find the area. S1 also placed the string horizontally with a length of 4 cm and vertically with a length of 3 cm without adding an extra square unit vertically to find the perimeter. S1 placed origami paper covering 4 square units horizontally and 3 square units vertically to find the area (Figure 3). S1 began to form conjectures using manipulative objects. However, the results differed from the statement on the task sheet. Due to difficulties, S1 calculated the perimeter and area of the rectangle manually using formulas. Based on the calculation, S1 stated that the result did not match the statement on the task sheet, and the counter-example presented posed a challenge for S1. We asked, "Was your conjecture wrong?" S1 replied that the calculation was correct, but the result differed from the initial conjecture.

On the other hand, S2 completed the proof task using a representation with manipulative objects, string, and origami paper (Figure 3). S2 placed the string horizontally with a length of 5 cm and vertically with a length of 2 cm correctly to find the perimeter. Then, S2 placed origami paper covering 5 square units horizontally and 2 square units vertically to find the area. Similarly, when finding the perimeter of the second rectangle, S2 placed the string horizontally with a length of 4 cm and vertically with a length of 3 cm correctly. Then, S2 found the area using origami paper by placing 4 square units horizontally and 3 square units vertically. At this stage, S2 was better at placing the string and origami paper according to the length and width of the rectangle. Realizing that the result differed from the statement on the task sheet, S2 proposed a new conjecture. S2 stated, "In some cases, the size of the area of a rectangle will affect its perimeter. However, in this case, increasing the area does not cause the perimeter to increase." The counter-example provided by S1 and S2 at the proof stage was still imperfect. However, S2 demonstrated that the counter-example was useful for refining and modifying the primitive conjecture. Further steps are needed to achieve a more comprehensive conjecture.

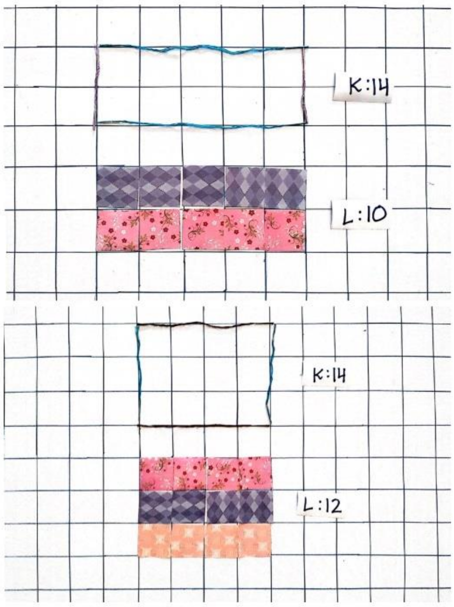

Figure 3. *S1 & S2 Faced with a Counter-Example*

Re-Examining Conjectures and Proofs

In this stage, students were asked to re-examine the conjectures they had made, leading them to realize that the counter-example they had proven was incomplete. We asked S1 and S2 to recheck the calculations of two rectangles, the first with a length of 5 cm and a width of 2 cm, and the second with a length of 4 cm and a width of 3 cm. From the action proof they created during the counter-example confrontation stage (see Figure 3), we asked the students to look at the number of square units in the first and second rectangles. Was there any difference in shape and area when viewed from the origami paper used? The students responded that the resulting shape was still a rectangle and that the amount of origami paper used increased when finding the area of the rectangle. However, when finding the perimeter, the length of string needed remained the same. Next, the researcher guided the students to calculate the area and perimeter of the two rectangles using formulas. The students stated that the results of the calculations using the formulas were the same as the results of the calculations using the concrete objects. Through this stimulation, the students formed new conjectures to reach a more comprehensive conjecture, stating, "In some cases, the perimeter of a rectangle does not increase even when the area of the rectangle is enlarged." The new conjectures of S1 and S2 were correct: "If the area of a rectangle is increased, the perimeter may remain the same or may increase" (Figure 3). A counter-example is used to illustrate the importance of the hypothesis of a theorem [13]. n this case, the conjecture met the conditions, but the conclusion was different.

IV. DISCUSSION

Based on the results described above, when students attempted to prove their primitive conjectures, they made conjectures that were still incorrect. Therefore, a confronted counter-example stage was necessary. The researcher provided a counter-example stimulation that was useful for discovering new conjectures and altering their primitive conjectures. At this stage, students experienced difficulties and confusion in completing the action proof when faced with the counter-example. However, the students continued to try to improve their primitive conjectures into new, though not yet comprehensive, conjectures. This indicates that students still struggled to complete the proof at certain stages [19]. The difficulty in completing the action proof occurred because the students found that their conjectures did not match the statement when faced with contradictory examples.

During the stage of proving their primitive conjecture, the action proof carried out by the students was reflected when they conducted the proof process using manipulative objects. The manipulative objects used in this study included string, origami paper, and white manila paper to represent the area and perimeter of a rectangle. The manipulative object provided by S1 was less accurate in representing the width of the rectangle. Additionally, S1 confirmed the statement that "when you increase the area of a rectangle, the perimeter always increases. Explain why or whether this is true? Is this statement true for all cases?" This means that the primitive conjecture they provided was incorrect. The students still failed to use logical manipulative objects [20]. This occurred because the conjectures they provided yielded results consistent with the statement, leading to their conjecture matching the statement. Since their conjecture was still incorrect at this stage, further proof at the next stage, which involves action proof, was required.

In the confronted counter-example stage, the students were faced with a counter-example. Through this stimulation, they realized that their initial conjecture, which they had previously proven, was still incorrect. Providing the counter-example during the action proof was done to stimulate the students to recognize the errors in their initial conjecture and to justify it into a more comprehensive mathematical conjecture [11]. Thus, the students attempted to make a new conjecture that "in some cases, the size of the area of a rectangle will affect its perimeter." The students showed the effect of the counter-example stimulation, although they had corrected the new conjecture, it was still imperfect. The counter-example stimulation provided did not make them give up easily. This stage demonstrated that counter-examples play a significant role in verifying a statement that can change thinking or methods to help reconsider their initial perception or conjecture [13]. The imperfection of the new conjecture occurred because the students still did not correctly represent the manipulative objects. Therefore, the third stage was necessary to achieve a more comprehensive conjecture.

In the stage of re-examining the conjecture with proof, the students completed the new, more comprehensive conjecture. Using the results of the action proof they created during the counter-example completion stage, the researcher provided verbal stimulation to guide them towards the correct answer. The students actively refined the conjecture, resulting in a more comprehensive new conjecture. Although the manipulative objects were not effective in their application, with the help of these manipulative objects, the conjecture they provided could be presented in a concrete manner, allowing for the verification of their primitive and new conjectures. The use of manipulative objects did not yet have a significant impact, but these objects provided authentic or tangible evidence that was easily accepted by elementary school students [21]. There was a difference between the mathematical proof completion by S1 and S2. S2 understood the concept and the use of concrete objects in representing each stage of the proof better than S1. Meanwhile, S1 was still confused in applying the manipulative objects and was more comfortable with the abstract methods typically used in the classroom. Consequently, S2 provided better action proof with manipulative objects than S1. This was likely due to S2's higher level of ability in solving problems compared to S1. Given S1's potential and persistence in solving problems, it is possible that, in time, S1 could reach the same level of ability as S2. With the support of educators and high-quality education, S1's academic and cognitive development could be positively influenced [22].

V. Conclusion

The action proof stages through counter-example stimulation showed improvement for students with both low and high levels of problem-solving abilities, leading towards more comprehensive conjectures and proof of correctness. Students with lower problem-solving abilities also showed improvement in understanding concepts and using manipulative objects. During the stage of proving primitive conjectures, both high- and low-ability students made primitive conjectures using manipulative objects. However, the primitive conjectures they provided were still incorrect. In the conjecture proof stage, both high- and low-ability students had conjectures that held the same meaning. At the stage of providing examples, both low- and high-ability students offered conjectures using manipulative objects, demonstrating improvement, though not yet comprehensive. In the stage of re-examining conjectures and proof, students were encouraged to refine their conjectures using manipulative objects into more comprehensive ones.

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